MAT 219 (lecture 11) part 1

§7.8 Repeated Eigenvalues

(These notes contain many extra examples which were not given during lecture.)

Next: §7.9 Nonhomogeneous Systems (End of Chapter 7!)

§7.8 Repeated Eigenvalues

EX: $\underline{x}' = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \underline{x}$

Eigenvalues $\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$ $\quad \boxed{\lambda = 3, 3}$

(Maybe $\lambda = 3$ will have two "fundamentally different" eigenvectors?)

(i.e. not multiples ~ a.k.a. "independent")

3-Eigenvector $\begin{bmatrix} 5-3 & -1 \\ 4 & 1-3 \end{bmatrix} \underline{v} = \underline{0}$

$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \boxed{\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

(The only possible solutions are multiples of this...)

General Solution

$$\underline{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(\underline{???} \right)$$

(There must be another fundamental solution!)

Outline

- Generalized eigenvectors
- e^{At} for repeated roots
- General formula for solution

Generalized Eigenvectors

In the example $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$ had repeated eigenvalue $\lambda = 3$ but only one 3-eigenvector $\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Basic Fact: Since $\lambda = 3$ came from $(\lambda - 3)^2 = 0$

We should have been solving

$$\begin{bmatrix} 5-3 & -1 \\ 4 & 1-3 \end{bmatrix}^2 \underline{v} = \underline{0}$$

(This is the "Cayley - Hamilton Theorem")

More generally, for λ repeated n times should solve $(A - \lambda I)^n \underline{v} = \underline{0}$

Two ways to have $\begin{bmatrix}5-3 & -1\\ 4 & 1-3\end{bmatrix}^2 \underline{v} = \underline{0}$!!

(1) $\begin{bmatrix}5-3 & -1\\ 4 & 1-3\end{bmatrix} \underline{v} = \underline{0}$

→ $\underline{v}$ is a 3-eigenvector
(all multiples $k\underline{v}$ will work)

(2) $\begin{bmatrix}5-3 & -1\\ 4 & 1-3\end{bmatrix} \underline{w} = \underline{v}$ where $\begin{bmatrix}5-3 & -1\\ 4 & 1-3\end{bmatrix} \underline{v} = \underline{0}$

→ $\underline{w}$ is a $\underline{v}$-generalized 3-eigenvector
(all sums $\underline{w} + k\underline{v}$ will work)

EX: $A = \begin{bmatrix}5 & -1\\ 4 & 1\end{bmatrix}$ has $\lambda = 3, 3$ w/ 3-eigenvect. $\underline{v} = \begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix}$-gen. eigenvect. $\begin{bmatrix}5-3 & -1\\ 4 & 1-3\end{bmatrix} \underline{w} = \begin{bmatrix}1\\2\end{bmatrix}$

$$\begin{cases} 2x - y = 1 \\ 4x - 2y = 2 \end{cases}$$

We want a simple nonzero solution to this. Since it is not $=0$ we can plug in $x=0$

Note: These two equations are multiples because there should be a line of solutions $\underline{w} + k\underline{v}$

→ let $x=0$ & solve for y:

$0 - y = 1 \rightarrow y = -1$

$$\boxed{\underline{w} = \begin{bmatrix}0\\-1\end{bmatrix}}$$

(Check: $\begin{bmatrix}2 & -1\\ 4 & -2\end{bmatrix}^2 \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}2 & -1\\ 4 & -2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix}0\\0\end{bmatrix}$)

Note: Plugging in other x-values gives $\underline{w} = \begin{bmatrix}0\\-1\end{bmatrix} + k\begin{bmatrix}1\\2\end{bmatrix}$

Basic Property of Eigenvectors:

$$A\underline{v} = \lambda\underline{v}$$

Basic Property of Generalized Eigenvectors:

$$A\underline{w} = \lambda\underline{w} + \underline{v}$$

EX: Find eigenvalues, eigenvectors, gen. eigenvect of $A = \begin{bmatrix}5 & 9\\ -1 & -1\end{bmatrix}$

Eigenvalues $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$ $\boxed{\lambda = 2, 2}$

2-eigenvect $\begin{bmatrix}5-2 & 9\\ -1 & -1-2\end{bmatrix} \underline{v} = \underline{0} \rightarrow \underline{v} = \begin{bmatrix}9\\-3\end{bmatrix} \xrightarrow{\times \frac{1}{3}} \boxed{\begin{bmatrix}3\\-1\end{bmatrix}}$

$\begin{bmatrix}3\\-1\end{bmatrix}$-gen. eigenv. $\begin{bmatrix}5-2 & 9\\ -1 & -1-2\end{bmatrix} \underline{w} = \begin{bmatrix}3\\-1\end{bmatrix}$

$$\begin{cases} 3x + 9y = 3 \\ -x - 3y = -1 \end{cases}$$

plug in $x=0$ & solve
$0 + 9y = 3 \rightarrow y = 1/3$

$$\boxed{\underline{w} = \begin{bmatrix}0\\1/3\end{bmatrix}}$$

Note: Cannot multiply $\underline{w}$ by 3 to change $\begin{bmatrix}0\\1/3\end{bmatrix} \xrightarrow{\times 3} \begin{bmatrix}0\\1\end{bmatrix}$

→ Instead we could

(1) Multiply $\underline{v}$ & $\underline{w}$ by 3

(2) Change $\underline{w}$ to $\begin{bmatrix}0\\1/3\end{bmatrix} + 1/3\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

Equivalent to setting $x=1$ & solving for y

EX: Find eigenvalues, eigenvectors, gen. eigenvectors of $A=\begin{bmatrix}-5 & 3\\ -3 & 1\end{bmatrix}$

Eigenvalues $\lambda^2+4\lambda+4=0$

$(\lambda+2)^2=0$ $\boxed{\lambda=-2,-2}$

(-2)-eigenvect $\begin{bmatrix}-5-(-2) & 3\\ -3 & 1-(-2)\end{bmatrix}\underline{v}=\underline{0} \rightsquigarrow v=\begin{bmatrix}3\\-3\end{bmatrix} \xrightarrow{\times\frac{1}{3}} \boxed{\begin{bmatrix}1\\-1\end{bmatrix}}$

$\begin{bmatrix}1\\-1\end{bmatrix}$-gen. eigenv $\begin{bmatrix}-5-(-2) & 3\\ -3 & 1-(-2)\end{bmatrix}\underline{w}=\begin{bmatrix}1\\-1\end{bmatrix}$

$\begin{bmatrix}3x+3y=1\\ -3x-3y=-1\end{bmatrix}$ let $x=0$ & solve

$0+3y=1 \rightsquigarrow y=\frac{1}{3}$

$w=\begin{bmatrix}0\\ \frac{1}{3}\end{bmatrix}$

So $\begin{bmatrix}1\\-1\end{bmatrix}$-gen. eigenv. is $w=\begin{bmatrix}0\\ \frac{1}{3}\end{bmatrix}$

$\times 3$

$\boxed{\begin{bmatrix}3\\-3\end{bmatrix}\text{-gen eigenv. is } w=\begin{bmatrix}0\\1\end{bmatrix}}$

Note: Generalized eigenvectors only exist when you need them.

→ If λ is not a repeated eigenvalue then there are no generalized eigenvectors

EX: Does $A=\begin{bmatrix}-1 & 2\\ 2 & 2\end{bmatrix}$ have generalized eigenvectors?

Eigenvalues: $\lambda^2-\lambda-6=0$

$(\lambda-3)(\lambda+2)=0$ $\boxed{\lambda=-2,3}$

(-2)-eigenvect: $\begin{bmatrix}-1-(-2) & 2\\ 2 & 2-(-2)\end{bmatrix}\underline{v}=\underline{0} \rightsquigarrow \underline{v}=\boxed{\begin{bmatrix}2\\-1\end{bmatrix}}$

~~$\begin{bmatrix}2\\-1\end{bmatrix}$-gen. eigenv.? $\begin{bmatrix}-1-(-2) & 2\\ 2 & 2-(-2)\end{bmatrix}\underline{w}=\begin{bmatrix}2\\-1\end{bmatrix}$~~

~~$\begin{bmatrix}x+2y=2\\ 2x+4y=-1\end{bmatrix}$~~ [This system has no solution – these are parallel lines!!]

3-eigenvect. $\begin{bmatrix}-1-3 & 2\\ 2 & 2-3\end{bmatrix}\underline{v}=\underline{0} \rightsquigarrow \underline{v}=\begin{bmatrix}2\\4\end{bmatrix} \xrightarrow{\times\frac{1}{2}} \boxed{\begin{bmatrix}1\\2\end{bmatrix}}$

~~$\begin{bmatrix}1\\2\end{bmatrix}$-gen. eigenv.? $\begin{bmatrix}-1-3 & 2\\ 2 & 2-3\end{bmatrix}\underline{w}=\begin{bmatrix}1\\2\end{bmatrix}$~~

~~$\begin{bmatrix}-4x+2y=1\\ 2x-y=2\end{bmatrix}$~~ [This system has no solution either!!]

Note: Sometimes repeated root problems don't need generalized eigenvect.

→ Sometimes you can get multiple eigenvectors with a single eigenvalue.

EX: Find eigenvalues, etc of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvalues: $(\lambda - 2)^2 = 0$ $\boxed{\lambda = 2, 2}$

2-eigenvect: $\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \underline{v} = \underline{0}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{v} = \underline{0}$

Note: cannot take any negative reciprocals!

- plug in $\begin{cases} x = 1 \\ y = 0 \end{cases}$ $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- plug in $\begin{cases} x = 0 \\ y = 1 \end{cases}$ $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

These are both nonzero solutions and are not multiples

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are "independent" eigenvectors

~~$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$-gen. eigenv.? $\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?~~

~~$\begin{bmatrix} 0 = 1 \\ 0 = 0 \end{bmatrix}$~~ No solution!

~~$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$-gen. eigenv.? $\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?~~

~~$\begin{bmatrix} 0 = 0 \\ 0 = 1 \end{bmatrix}$~~ No solution!

→ Please only look for generalized eigenvectors when you know they should exist. (i.e. only if you are missing eigenvectors.)

3x3 case is more interesting.

If A is 3x3 then λ could be repeated 3 times!

→ In this case, you should solve

$$(A - \lambda I)^3 \underline{v} = \underline{0}$$

→ "Chains" of generalized eigenvectors

1. $(A - \lambda I)\underline{v} = \underline{0}$

   $\underline{v}$ is a λ-eigenvector $(A\underline{v} = \lambda \underline{v})$

2. $(A - \lambda I)\underline{w} = \underline{v}$ with $(A - \lambda I)\underline{v} = \underline{0}$

   $\underline{w}$ is a $\underline{v}$-generalized λ-eigenvector $(A\underline{w} = \lambda \underline{w} + \underline{v})$

3. $(A - \lambda I)\underline{u} = \underline{w}$ with $(A - \lambda I)\underline{w} = \underline{v}$ etc.

   $\underline{u}$ is a $\underline{w}$-generalized, $\underline{v}$-generalized λ-eigenvector $(A\underline{u} = \lambda \underline{u} + \underline{w})$

Note: Sometimes you may have

- 1 eigenvector + 1 gen. eigenv. + 1 gen.^{2} eigenv

or

- 2 eigenvectors + 1 gen. eigenv

or

- 3 eigenvectors

EX: Find eigenvectors, etc of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix} \text{ with } \lambda = 2, 2, 2$$

2-eigenvect $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 0 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x &= 0 \\ 3x - y &= 0 \end{aligned}\right.$

→ Cannot plug in $x=1$ because of 2^{nd} eqn.

→ Cannot plug in $y=1$ because then $\begin{cases} x = 0 \\ 3x - 1 = 0 \end{cases}$

→ Plug in $\underline{\underline{z=1}}$ & solve $\underline{\underline{x=0}}$

$\Rightarrow 0 - y = 0 \rightsquigarrow \underline{\underline{y=0}}$

$\boxed{\underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$-gen. eigenv $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 0 \end{bmatrix} \underline{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x &= 0 \\ 3x - y &= 1 \end{aligned}\right.$

Plug in $z=1$ above ⟶ Plug in $\underline{\underline{z=0}}$ now

$\underline{\underline{x=0}}$

$0 - y = 1 \rightsquigarrow \underline{\underline{y=-1}}$

$\boxed{\underline{w} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}$

$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$-$\text{gen}^2$ eigenv $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 0 \end{bmatrix} \underline{u} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x &= -1 \\ 3x - y &= 0 \end{aligned}\right.$

Plug in $z=1$ for $\underline{v}$ ⟶ Plug in $\underline{\underline{z=0}}$ now

$\underline{\underline{x=-1}}$

$-3 - y = 0 \rightsquigarrow \underline{\underline{y=-3}}$

$\boxed{\underline{u} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}}$

2-eigenv. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has gen. eigenv $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ has gen. eigenv. $\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$

EX: Find eigenvectors, etc of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \text{ with } \lambda = 1, 1, 1$$

1-eigenvect: $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \left[\begin{aligned} 2y &= 0 \\ x + 2z &= 0 \\ -y &= 0 \end{aligned}\right.$

→ Eqn 1 & 3 ⇒ $\underline{\underline{y=0}}$

plug in $\underline{\underline{x=1}}$ & solve $\underline{\underline{z=-\tfrac{1}{2}}}$

$\underline{v} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \xrightarrow{\times 2} \boxed{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}$

$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$-gen. eigenv: $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \underline{w} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \rightsquigarrow \left[\begin{aligned} 2y &= 2 \\ x + 2z &= 0 \\ -y &= -1 \end{aligned}\right.$

Plug in $x=1$ for $\underline{v}$ ⟶ Plug in $\underline{\underline{x=0}}$ now

$2y = 2 \rightsquigarrow \underline{\underline{y=1}}$

$0 + 2z = 0 \rightsquigarrow \underline{\underline{z=0}}$

$\boxed{\underline{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$-$\text{gen}^2$ eigenv: $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \underline{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{aligned} 2y &= 0 \\ x + 2z &= 1 \\ -y &= -1 \end{aligned}\right.$

Plug in $x=1$ for $\underline{v}$ ⟶ Plug in $\underline{\underline{x=0}}$ now

$2y = 0 \rightsquigarrow \underline{\underline{y=0}}$

$0 + 2z = 1 \rightsquigarrow \underline{\underline{z=\tfrac{1}{2}}}$

$\boxed{\underline{u} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}}$

1-eigenv. $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ has gen eigenv $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has gen eigenv $\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$

$\times 2$ (Multiply entire chain)

1-eigenv. $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ has gen eigenv $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ has gen eigenv $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Note: Sometimes you don't need gen^2 eigenvectors

EX: Find eigenvectors etc of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \text{ with } \lambda = 1, 1, 1$$

1-eigenvector

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x - y + z &= 0 \\ x - y + z &= 0 \end{aligned}\right.$$

→ plugging in $x = 1$ is not enough to determine soln.

Two vectors: (1) plug in $\begin{cases} x = 1 \\ y = 0 \end{cases}$ & solve for z:

$1 - 0 + z = 0 \rightsquigarrow \underline{z = -1}$

(2) plug in $\begin{cases} x = 0 \\ y = 1 \end{cases}$ & solve for z:

$0 - 1 + z = 0 \rightsquigarrow \underline{z = 1}$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \& \quad \underline{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \text{ are eigenvect.}$$

gen. eigenv:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \underline{w} = \overbrace{a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{generalizes a linear combination...}} \rightsquigarrow \left[\begin{aligned} 0 &= a \\ x - y + z &= b \\ x - y + z &= -a + b \end{aligned}\right.$$

→ Eqn 1 says $a = 0$, so we can pick $b = 1$

Plug in for x & y for $\underline{v}$ ⟹ plug in $\begin{cases} x = 0 \\ y = 0 \end{cases}$ now

$0 - 0 + z = 1 \rightsquigarrow \underline{z = 1}$

$\underline{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is gen eigenv. of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

The final step here can get complicated...

EX: Find eigenvectors etc of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \text{ with } \lambda = 2, 2, 2$$

2-eigenvector

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x &= 0 \\ 3x &= 0 \end{aligned}\right.$$

→ Cannot plug in $x = 1$

→ Plugging in $y = 1$ is not enough to determine z.

Two vectors: (1) Plug in $\begin{cases} y = 1 \\ z = 0 \end{cases}$ & solve for x:

$\underline{x = 0}$

(2) Plug in $\begin{cases} y = 0 \\ z = 1 \end{cases}$ & solve for x:

$\underline{x = 0}$

$$\boxed{\underline{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \text{ are eigenvect.}$$

gen. eigenvect:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \underline{w} = a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{aligned} 0 &= 0 \\ x &= a \\ 3x &= b \end{aligned}\right.$$

→ Must have $b = 3a$. Let $a = 1$

$\Rightarrow b = 3$.

Plugged in for y & z for $\underline{v}$ ⟹ plug in $\begin{cases} y = 0 \\ z = 0 \end{cases}$ now

$\rightsquigarrow \underline{x = 1}$

$\underline{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is gen. eigenvector of $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Other good examples (Homework):

- $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} 7 & 4 \\ -4 & -1 \end{bmatrix}$

- $A = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} 9 & -9 \\ 1 & 3 \end{bmatrix}$

- $A = \begin{bmatrix} -10 & 4 \\ -9 & 2 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ $\lambda = 2, 2, 2$

- $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $\lambda = 1, 1, 1$

- $A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 2 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ $\lambda = \underline{\underline{2}}, \underline{\underline{3, 3}}$

- $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$ $\lambda = 1, 1, 1$

- $A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$ $\lambda = -1, -1, -1$

- $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 5 & -2 \\ 0 & 1 & 2 \end{bmatrix}$ $\lambda = 3, 3, 3$